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Hydrodynamic interactions of spherical particles in quadratic Stokes flows

S. Haber[a], H. Brenner[b,*]

[a]Faculty of Mechanical Engineering, Technion, Haifa 32000, Israel
[b]Department of Chemical Engineering, MIT, Cambridge, MA 02139-4307, USA

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This paper is dedicated to Gad Hetsroni, mentor, friend and colleague, on the occasion of his 65th birthday. Although not present at your actual birth, one of us was nevertheless present at the birth of the *International Journal of Multiphase Flow* — a journal which bears your indelible stamp.

Abstract

The quasi-steady hydrodynamic Stokes drag force and torque exerted on each of N non-identical particles immersed in a general quadratic undisturbed flow field at infinity is analytically investigated. Explicit results are given for the case of two spherical particles. In particular, expressions are obtained for these forces and torques in terms of: (1) the linear and angular velocities of each of the particles together with the local shear rate of the quadratic flow, all evaluated at the center of volume of the two-sphere system; and (2) the position-independent triadic shear-rate gradient characterizing the undisturbed quadratic flow field. These tensorial (matrix) formulas, obtained by the method of reflections for the intrinsic Stokes resistance coefficients for each of the spheres, are expressed nondimensionally in terms of the respective sphere radii, the center-to-center distance $2H$ between spheres, a unit vector $\mathbf{e}$ lying along the line-of-centers, and a characteristic length-scale R_0 of the undisturbed quadratic flow field — typically the radius of a tube bounding an axisymmetric Poiseuille flow in which the two spheres are suspended. The reflection scheme is accurate within an error of $O[(r/H)^{\alpha}(r/R_0)^{\beta}]$, wherein $\alpha+\beta \leq 6$ and in which r is a characteristic sphere radius. The respective translational velocities of the two spheres are derived for the particular case where each is freely suspended in an unbounded Poiseuille flow (i.e., wall effects are neglected). It is shown in this case that a net radial motion of the pair of neutrally buoyant spheres ensues across the undisturbed streamlines as a consequence of the quadratic nature of the flow field (coupled with the interparticle hydrodynamic interaction). The direction of net migration calculated for the two spheres is from low to high shear

* Corresponding author. Tel.: +1-617-253-6687; Fax: +1-617-258-8224.
E-mail address: hbrenner@mit.edu (H. Brenner)

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rates. This contrasts with shear-induced migration phenomena observed in concentrated suspensions subjected to *inhomogeneous* shearing flows, where the direction of particle migration is from high to low shear rates. The two opposing effects acting in concert may help explain how a *steady-state* radial particle concentration distribution can be achieved for suspensions of non-Brownian particles.

Keywords: Low Reynolds number hydrodynamics; Hydrodynamic interaction of particles; Particulate Stokes flow

1. Introduction

Much attention has been focused in recent years on the behavior of suspensions subjected to *inhomogeneous* shear flows, and in particular on shear-induced, cross-streamline particle migration phenomena. Experimental measurements by Karnis et al. (1966) revealed that (neutrally-buoyant particle) velocity profiles in dense two-phase flows differ markedly from those observed for single-phase flows, even at very small Reynolds numbers. Eckstein et al. (1977) and later Leighton and Acrivos (1987a, 1987b) introduced the kinematical notion of a shear-induced particulate self-diffusivity coefficient in an attempt to quantify the observed cross-streamline migration phenomena. Gadala-Maria and Acrivos (1980) observed that viscous resuspension of the particles occurred during the course of their measurements of the rheological properties of suspensions of coal particles, a fact that could possibly rationalize otherwise unexplained rheological differences observed during comparable homogeneous shear flow experiments performed on these same suspensions. Enhanced experiments were later carried out by others, including Koh et al. (1994), Averbakh et al. (1997), and Shauly et al. (1997) using laser-Doppler velocimetry. Their reports provide rich and detailed data on velocity profiles and local particle concentration distributions in suspensions, thereby furnishing explicit visual evidence of the lateral migration phenomena whose existence was only implicitly inferred in prior work. Again, non-uniform radial particle concentration profiles were attributed to shear-induced lateral migration phenomena.

Various mechanisms have been postulated attempting to explain why — at low Reynolds numbers, where Segre–Silberberg-type inertial effects are absent — freely suspended particles migrate across the streamlines of the ‘undisturbed’ flow, a phenomenon that is impossible for a single, freely suspended sphere, even in the presence of wall effects (Happel and Brenner, 1983). It is now well accepted that hydrodynamic interactions among the suspended particles constitutes the most likely mechanism responsible for the phenomenon of radial migration.

Since analytical solutions of multiparticle Stokes-flow problems are intractable, three basic approaches have been used in theoretical attempts to quantify the observed phenomena: (1) analytical/numerical approaches towards solving the underlying multiparticle hydrodynamic equations, such as Stokesian dynamics (Brady, 1988). In addition, related techniques (e.g., Hassonjee et al., 1988, 1992; Brenner et al., 1990; Kim and Karrila, 1991; Chang and Powell, 1993; Nott and Brady, 1994) have been employed, wherein the individual motions of a large number of suspended particles were followed using either collocation methods or boundary integrals. However, because of the use of spatially-periodic boundary conditions or cell models in these approaches, such schemes appear ill suited to treating unbounded *inhomogeneous* flow problems of the present type; (2) a second purely phenomenological, kinematical approach, employed by Leighton and Acrivos (1987b), Phillips et al. (1992), and others, proved to be quite successful in reproducing many of the features observed during narrow-gap Couette experiments (Acrivos et al., 1993). The scheme involves introducing an empirical shear-induced self-diffusion coefficient based on scaling arguments, subsequently using available experimental data to quantify the appropriate parameters appearing in their phenomenological description; (3) the third approach, which is applicable to the entire class of multiparticle Stokes-flow problems, uses degenerate microscale models of local particle conformations in suspensions to approach analytically this class of hydrodynamic interaction problems. The latter schemes aim to encapsulate the primary effects arising from hydrodynamic interactions between a small number of particles, while remaining sufficiently simple in scope to be handled analytically in many cases. Results obtained using this in-depth view of systems containing a small number of interacting particles provide a rational vehicle for deriving the macroscopic properties of suspensions, at least in the dilute particle limit. It enables one to predict not only the self-diffusion phenomena in cases where the shear rate is spatially uniform, but also enables calculating the effects of *varying* concentrations on radial migration (termed transverse, shear-induced, gradient diffusion). Spatially periodic as well as two- or three-body configurations were studied. For example, Zuzovski et al. (1983) and Adler et al. (1985) used a spatially-periodic model to obtain the rheological properties of suspensions. Batchelor (1972) used two-body interactions to calculate the sedimentation velocity of a dilute suspension of rigid non-neutrally-buoyant spherical particles, whereas Batchelor and Green (1972b) employed two-body interactions to calculate the rheological properties of dilute suspensions. Haber et al. (1990) used two-body interactions to derive the dispersion coefficient of a dilute suspension of flexible dumb-bells. Wang et al. (1998) invoked three-body interactions to calculate radial particle diffusivities in dilute suspensions undergoing shear. Many other applications of the general philosophy embodied in (3) exist in the literature.

As indicated above, the mainstay of past analytical investigations of dilute suspensions has involved taking account of the hydrodynamic interactions between two or three rigid spheres, suspended either in quiescent fluids or in fluids undergoing simple shear flow. Several of these were investigated analytically, either exactly via use of bipolar coordinates or approximately by the method of reflections. Other schemes utilized purely numerical methods, such as collocations or boundary integrals. Stimson and Jeffery (1926) were the first to obtain an analytical expression for the drag force exerted on two identical spheres moving along their line-of-centers through a quiescent fluid. Brenner and O'Neill (1972) as well as Batchelor and Green (1972a) calculated the hydrodynamic interactions between two identical spheres in linear shear flows. Jeffrey and Onishi (1984) derived the hydrodynamic interactions between two moving non-identical spheres. Kim (1987) calculated three-body interactions. Jeffrey (1992) obtained the resistance matrices for two unequal spheres immersed in a general homogeneous shear flow. Most recently, Cichocki et al. (1994) obtained the drag forces exerted on a large number of spherical particles comprising part of a suspension. Although a vast number of such theoretical studies exist for two or three spheres immersed in either quiescent or homogeneously-sheared fluids, to the best of our knowledge no analytical results exist for the case of two spheres undergoing *inhomogeneous* shear flow, particularly quadratic flows.

Motivation for studying the latter classes of flows is provided by ongoing interest in seeking a rational explanation of lateral migration phenomena.

Wang et al. (1998), who investigated three-body systems, proved that two-body interactions in simple *shear* flow fail to predict transverse shear-induced gradient diffusion. It has not been investigated, however, whether a *two*-body system immersed in an *inhomogeneous* shear flow, e.g., a quadratic flow, can give rise to permanent lateral migration and, hence, to transverse 'diffusion.' It is well known (see e.g., Happel and Brenner, 1983) that in the absence of true wall effects (the center of) a *single*, freely suspended sphere adopts the undisturbed local fluid velocity in its proximity in a simple shear flow, but lags behind the local Poiseuille parabolic velocity in the quadratic flow occurring in a circular tube. In neither case, does such an isolated sphere undergo radial migration, a conclusion which applies even when true wall effects are taken into account.

The objective of this paper is to broadly outline a generic approach to two-sphere hydrodynamic interaction problems in inhomogeneous flow fields — more specifically to investigate the effect of *general* quadratic flows on the motions of each of two neutrally buoyant spheres in the quasi-static Stokes flow regime. Initially, we develop a general theory for the hydrodynamic forces and torques on an N-sphere system in a quadratic flow field at infinity when each of the spheres translates and rotates with arbitrarily prescribed velocities. From this analysis, we show how to extract the contribution of the quadratic flow to the intrinsic hydrodynamic resistance coefficients from the far simpler boundary-value problem, wherein one of the spheres moves through an otherwise *quiescent* fluid, i.e., one in which the fluid at infinity is at rest as too are the other spheres. As a special case the hydrodynamic resistance tensors are calculated for two spheres immersed in a general quadratic flow. We then examine whether the hydrodynamic drag forces exerted on two-freely suspended spheres in an unbounded Poiseuille flow will result in permanent lateral migration of the pair, albeit in the absence of wall effects. The results thereby obtained are expected to facilitate better understanding of the mechanisms contributing to shear-induced gradient diffusion phenomena.

2. Statement of the problem

Consider N non-identical rigid spherical particles of respective radii r_I $(I = 1, \ldots, N)$ suspended in an incompressible fluid of viscosity μ which is subjected to a general unbounded quadratic flow field $\mathbf{v}^\infty$. The position vector of the center $\mathbf{R}^{OI}$ of the Ith particle is measured from an arbitrary reference point O fixed in the fluid. The center of the Ith particle moves instantaneously with velocity $\mathbf{V}^I$, while the particle rotates with angular velocity $\boldsymbol{\Omega}^I$. The undisturbed fluid velocity field measured with respect to an origin situated at O by an observer at rest is given by the general expression

$$\mathbf{v}^\infty = \mathbf{v}^O + \boldsymbol{\omega}^O \times \mathbf{x}^O + \mathbf{S}^O \cdot \mathbf{x}^O + \mathbf{E}{:}\mathbf{x}^O\mathbf{x}^O, \tag{1}$$

where the radius vector $\mathbf{x}^O$ is measured from O. We have no need for the corresponding pressure field p^∞. The undisturbed approach velocity $\mathbf{v}^O$, angular velocity $\boldsymbol{\omega}^O$, and shear-rate dyadic $\mathbf{S}^O$ are all evaluated at O. Each depends upon the location of O. However, the shear-rate gradient triadic $\mathbf{E}$ is a constant of the flow field, independent of location. With no loss of generality, $\mathbf{S}^O$ is symmetric and $E_{ijk} = E_{ikj}$. (Henceforth, majuscule I, J, K superscripts will be used to refer to the particle number and minuscule indices i, j, k to tensorial components. Einstein's summation convention applies to the latter.) Since $\mathbf{v}^\infty$ is a solenoidal field, $S_{ii}^O = 0$ and $E_{iik} = 0$. Obviously, at a different reference point, say P, $\mathbf{v}^P$, $\boldsymbol{\omega}^P$ and $\mathbf{S}^P$ can easily be evaluated for a given vector displacement $\mathbf{a}^{OP}$ from O to P:

$$\mathbf{v}^P = \mathbf{v}^O + \boldsymbol{\omega}^O \times \mathbf{a}^{OP} + \mathbf{S}^O \cdot \mathbf{a}^{OP} + \mathbf{E}{:}\mathbf{a}^{OP}\mathbf{a}^{OP}, \tag{2a}$$

$$\boldsymbol{\omega}^P = \boldsymbol{\omega}^O + \frac{1}{2}\boldsymbol{\varepsilon}{:}\left[\mathbf{E}\cdot\mathbf{a}^{OP} - (\mathbf{E}\cdot\mathbf{a}^{OP})^{\mathrm{T}}\right], \tag{2b}$$

$$\mathbf{S}^P = \mathbf{S}^O + \left[\mathbf{E}\cdot\mathbf{a}^{OP} + (\mathbf{E}\cdot\mathbf{a}^{OP})^{\mathrm{T}}\right]. \tag{2c}$$

Here, $\boldsymbol{\varepsilon}$ is the permutation triadic, and $(\cdot)^{\mathrm{T}}$ is the RHS transposition operator. Obviously, for homogeneous shear fields, $\boldsymbol{\omega}$ and $\mathbf{S}$ are independent of location, since $\mathbf{E} = \mathbf{0}$.

Assuming creeping flow, the flow field in the presence of the spheres is governed by Stokes equations of motion:

$$\mu\nabla^2\mathbf{v} = \nabla p, \qquad \nabla\cdot\mathbf{v} = 0, \tag{3}$$

subject to the no-slip velocity conditions on all particle boundaries:

$$\mathbf{v} = \mathbf{V}^I + \boldsymbol{\Omega}^I \times \mathbf{x}^I \quad \text{on all } \partial S^I \ (I = 1,2,\ldots,N), \tag{4}$$

where ∂S^I denotes the boundary of the Ith sphere and $\mathbf{x}^I$ is the radius vector measured from the center of particle I. As $|\mathbf{x}|\to\infty$ the disturbance velocity generated by the motion of the particles vanishes, whence $\mathbf{v}\to\mathbf{v}^\infty$.

We seek to obtain expressions for the drag forces and torques exerted on each of the individual N spheres in terms of the translational and angular velocities of all of the spheres plus the undisturbed shear rate and shear-rate gradient. This information will subsequently enable us to obtain explicit results for the respective Stokes resistance tensors in the case of two-spheres immersed in a *general* quadratic flow. In turn, this will eventually furnish the translational and angular velocities of a pair of freely suspended spheres in an axisymmetric Poiseuille flow.

3. Analysis and results

3.1. N-particle system

Define a disturbance velocity field $\mathbf{u}$ and accompanying pressure field q as

$$\mathbf{v} = \mathbf{v}^\infty + \mathbf{u}, \qquad p = p^\infty + q. \tag{5}$$

Obviously, the fields ($\mathbf{u}$, q) are governed by the Stokes equations, vanish at infinity, and satisfy the respective boundary conditions

$$\mathbf{u}^I = (\mathbf{V}^I - \mathbf{v}^I) + (\boldsymbol{\Omega}^I - \boldsymbol{\omega}^I) \times \mathbf{x}^I - \mathbf{S}^I \cdot \mathbf{x}^I - \mathbf{E}{:}\mathbf{x}^I\mathbf{x}^I \quad \text{on } \partial S^I \ (I = 1,2,\ldots,N). \tag{6}$$

Note that $\mathbf{v}^\infty$ appearing above is not expressed in terms of a *single*, fluid-fixed, reference point, but is, rather, expressed in terms of respective reference points permanently fixed in each of the particles (from which points the respective position vectors $\mathbf{x}^I$ are measured). That is, in the explicit expression (6) for the boundary condition on the surface of the Ith sphere, the pertinent reference point lies at the center of that sphere. Thus, $\mathbf{S}^I$, $\mathbf{v}^I$ and $\boldsymbol{\omega}^I$ are the respective shear rate, translational and angular velocities of the undisturbed velocity field evaluated at the center of the Ith sphere. These velocities are, of course, known for a given instantaneous configuration of the N-particle system when the undisturbed flow field is specified. Due to the fact that $\mathbf{S}$ changes with location, the solution scheme used here represents a modification of that utilized by Brenner and O'Neill (1972) to derive the Stokes resistance tensors for multiparticle systems subjected to *linear* (i.e., homogeneous) shear fields.

Owing to linearity, the solution for ($\mathbf{u}$, q) can be expressed as the sum

$$\mathbf{u} = \sum_1^N \mathbf{u}^I, \qquad q = \mu \sum_1^N q^I, \tag{7}$$

where each of the fields ($\mathbf{u}^I$,q^I) is governed by the Stokes equations (3) (with $\mu = 1$), vanish at infinity, and satisfy the following boundary conditions:

$$\mathbf{u}^I = (\mathbf{V}^I - \mathbf{v}^I) + (\boldsymbol{\Omega}^I - \boldsymbol{\omega}^I) \times \mathbf{x}^I - \mathbf{S}^I \cdot \mathbf{x}^I - \mathbf{E}{:}\mathbf{x}^I\mathbf{x}^I \quad \text{on } \partial S^I \tag{8a}$$

and

$$\mathbf{u}^I = \mathbf{0} \quad \text{on all } \partial S^J \ (J \neq I). \tag{8b}$$

Again, owing to the linearity of the Stokes equations, the general solution for the ($\mathbf{u}^I$,q^I) fields and the corresponding stress fields $\boldsymbol{\pi}^I$ is

$$u_i^I = U_{ij}^{VI}\left(V_j^I - v_j^I\right) + U_{ij}^{\Omega I}\left(\Omega_j^I - \omega_j^I\right) + U_{ijk}^{SI} S_{jk}^I + U_{ijkl}^{EI} E_{jkl}, \tag{9a}$$

$$q^I = Q_j^{VI}\left(V_j^I - v_j^I\right) + Q_j^{\Omega I}\left(\Omega_j^I - \omega_j^I\right) + Q_{jk}^{SI} S_{jk}^I + Q_{jkl}^{EI} E_{jkl}, \tag{9b}$$

$$\pi_{ni}^I = \Pi_{nij}^{VI}\left(V_j^I - v_j^I\right) + \Pi_{nij}^{\Omega I}\left(\Omega_j^I - \omega_j^I\right) + \Pi_{nijk}^{SI} S_{jk}^I + \Pi_{nijkl}^{EI} E_{jkl}, \tag{9c}$$

where the tensor fields ($\mathbf{U}^{VI}$,$\mathbf{Q}^{VI}$), ($\mathbf{U}^{\Omega I}$,$\mathbf{Q}^{\Omega I}$), ($\mathbf{U}^{SI}$,$\mathbf{Q}^{SI}$), ($\mathbf{U}^{EI}$,$\mathbf{Q}^{EI}$) satisfy Stokes equations:

$$\frac{\partial^2 U_{ij}^{VI}}{\partial x_m \partial x_m} = \frac{\partial Q_j^{VI}}{\partial x_i}, \quad \frac{\partial U_{ij}^{VI}}{\partial x_i} = 0; \tag{10a}$$

$$\frac{\partial^2 U_{ij}^{\Omega I}}{\partial x_m \partial x_m} = \frac{\partial Q_j^{\Omega I}}{\partial x_i}, \quad \frac{\partial U_{ij}^{\Omega I}}{\partial x_i} = 0; \tag{10b}$$

$$\frac{\partial^2 U_{ijk}^{\Omega I}}{\partial x_m \partial x_m} = \frac{\partial Q_{jk}^{\Omega I}}{\partial x_i}, \quad \frac{\partial U_{ijk}^{\Omega I}}{\partial x_i} = 0; \tag{10c}$$

$$\frac{\partial^2 U_{ijkl}^{EI}}{\partial x_m \partial x_m} = \frac{\partial Q_{jkl}^{EI}}{\partial x_i}, \quad \frac{\partial U_{ijkl}^{EI}}{\partial x_i} = 0; \tag{10d}$$

and the boundary conditions

$$U_{ij}^{VI} = \left\{ \begin{matrix} \delta_{ij} \quad \text{on } \partial S^I, \\ 0 \quad \text{on } \partial S^J \; (J \neq I) \end{matrix} \right\}; \tag{11a}$$

$$U_{ij}^{\Omega I} = \left\{ \begin{matrix} \varepsilon_{ijk} x_k^I \quad \text{on } \partial S^I, \\ 0 \quad \text{on } \partial S^J \; (J \neq I) \end{matrix} \right\}; \tag{11b}$$

$$U_{ijk}^{SI} = \left\{ \begin{matrix} -\frac{1}{2}\left(\delta_{ij} x_k^I + \delta_{ik} x_j^I\right) \quad \text{on } \partial S^I, \\ 0 \quad \text{on } \partial S^J \; (J \neq I) \end{matrix} \right\}; \tag{11c}$$

$$U_{ijkl}^{EI} = \left\{ \begin{matrix} -\delta_{ij} x_k^I x_l^I \quad \text{on } \partial S^I, \\ 0 \quad \text{on } \partial S^J \; (J \neq I) \end{matrix} \right\}. \tag{11d}$$

In the foregoing equations, the respective contributions arising from each of the four terms appearing in Eq. (8a) are separately evaluated and the results superimposed. Thus, for instance, $\mathbf{U}^{VI}$ is affected only by the velocity difference $\mathbf{V}^I - \mathbf{v}^I$ upon setting $\boldsymbol{\Omega}^I - \boldsymbol{\omega}^I$, $\mathbf{S}^I$ and $\mathbf{E}$ equal to zero over ∂S^I (and over all other ∂S^J). The corresponding stress fields are

$$\Pi_{nij}^{VI} = -\delta_{ni} Q_j^{VI} + \frac{\partial V_{ij}^{VI}}{\partial x_n} + \frac{\partial V_{nj}^{VI}}{\partial x_i}, \tag{12a}$$

$$\Pi_{nij}^{\Omega I} = -\delta_{ni} Q_j^{\Omega I} + \frac{\partial V_{ij}^{\Omega I}}{\partial x_n} + \frac{\partial V_{nj}^{\Omega I}}{\partial x_i}, \tag{12b}$$

$$\Pi_{nijk}^{SI} = -\delta_{ni} Q_{jk}^{SI} + \frac{\partial V_{ijk}^{SI}}{\partial x_n} + \frac{\partial V_{njk}^{SI}}{\partial x_i}, \tag{12c}$$

$$\Pi_{nijkl}^{EI} = -\delta_{ni} Q_{jkl}^{EI} + \frac{\partial V_{ijkl}^{EI}}{\partial x_n} + \frac{\partial V_{njkl}^{EI}}{\partial x_i}. \tag{12d}$$

Hence, from Eqs. (10) and (11), it is obvious that the tensor fields $(\mathbf{U}^{VI},\mathbf{Q}^{VI},\boldsymbol{\Pi}^{VI})$, $(\mathbf{U}^{\Omega I},\mathbf{Q}^{\Omega I},\boldsymbol{\Pi}^{VI})$, $(\mathbf{U}^{SI},\mathbf{Q}^{SI},\boldsymbol{\Pi}^{SI})$, and $(\mathbf{U}^{EI},\mathbf{Q}^{EI},\boldsymbol{\Pi}^{EI})$ depend upon the configuration of the N-particle system and the respective sphere radii, but not on the fluid viscosity nor on the forcing terms $\mathbf{V}^I-\mathbf{v}^I,\boldsymbol{\Omega}^I-\boldsymbol{\omega}^I,\mathbf{S}^I$ and $\mathbf{E}$.

3.1.1. Hydrodynamic drag forces and torques

The hydrodynamic drag force $\mathbf{f}^I$ exerted on particle I and the torque $\mathbf{t}^I$ about the center of sphere I are obtained from the total contribution of all stress fields $\boldsymbol{\pi}^K$ $(K=1,2,\ldots,N)$:

$$\mathbf{f}^I=\mu\sum_{K=1}^{N}\int_{\delta S^I}\mathbf{n}\cdot\boldsymbol{\pi}^K\,\mathrm{d}S,\quad \mathbf{t}^I=\mu\sum_{K=1}^{N}\int_{\delta S^I}\mathbf{x}^I\times(\mathbf{n}\cdot\boldsymbol{\pi}^K)\,\mathrm{d}S. \tag{13}$$

Substituting Eq. (9c) into Eq. (13) yields

$$-\frac{1}{\mu}\mathbf{f}^I=\sum_{K=1}^{N}\mathbf{K}^{IK}\cdot(\mathbf{V}^K-\mathbf{v}^K)+\mathbf{D}^{IK}\cdot(\boldsymbol{\Omega}^K-\boldsymbol{\omega}^K)+\boldsymbol{\Phi}^{IK}:\mathbf{S}^K+\mathbf{F}^{IK}\vdots\mathbf{E}, \tag{14a}$$

$$-\frac{1}{\mu}\mathbf{t}^I=\sum_{K=1}^{N}\mathbf{C}^{IK}\cdot(\mathbf{V}^K-\mathbf{v}^K)+\mathbf{L}^{IK}\cdot(\boldsymbol{\Omega}^K-\boldsymbol{\omega}^K)+\boldsymbol{\Psi}^{IK}:\mathbf{S}^K+\mathbf{T}^{IK}\vdots\mathbf{E}. \tag{14b}$$

The triple-dot operation in Eq. (14), e.g., $\mathbf{F}^{IK}\vdots\mathbf{E}$, means that the ith component of the resulting vector is equal to $F^{IK}_{ijkl}E_{jkl}$. The drag resistance tensors $\mathbf{K}^{IK},\mathbf{D}^{IK},\boldsymbol{\Phi}^{IK},\mathbf{F}^{IK},\mathbf{C}^{IK},\mathbf{L}^{IK},\mathbf{D}^{IK},\boldsymbol{\Psi}^{IK},\mathbf{F}^{IK}$ and T^{IK} are obtained from the stress fields defined in Eq. (12) as:

$$K^{IK}_{ij}=-\int_{\partial S^I}\Pi^{VK}_{nij}n_n\,\mathrm{d}S,\qquad D^{IK}_{ij}=-\int_{\partial S^I}\Pi^{\Omega K}_{nij}n_n\,\mathrm{d}S, \tag{15a}$$

$$\Phi^{IK}_{ijk}=-\int_{\partial S^I}\Pi^{SK}_{nijk}n_n\,\mathrm{d}S,\qquad F^{IK}_{ijkl}=-\int_{\partial S^I}\Pi^{EK}_{nijkl}n_n\,\mathrm{d}S, \tag{15b}$$

$$C^{IK}_{ij}=-\varepsilon_{imp}\int_{\partial S^I}x^I_m\Pi^{VK}_{npj}n_n\,\mathrm{d}S,\qquad L^{IK}_{ij}=-\varepsilon_{imp}\int_{\partial S^I}x^I_m\Pi^{\Omega K}_{npj}n_n\,\mathrm{d}S, \tag{15c}$$

$$\Psi^{IK}_{ijk}=-\varepsilon_{imp}\int_{\partial S^I}x^I_m\Pi^{SK}_{npjk}n_n\,\mathrm{d}S,\ T^{IK}_{ijkl}=-\varepsilon_{imp}\int_{\partial S^I}x^I_m\Pi^{EK}_{npjkl}n_n\,\mathrm{d}S. \tag{15d}$$

Eq. (14) constitutes the general expression sought for the drag force and the torque exerted on any sphere in the system. It represents a generalization of the known case (Brenner and O'Neill, 1972) for linear undisturbed velocity fields, which being linear necessarily possess spatially uniform angular velocities and shear rates. Clearly, from Eqs. (10), (11), (12) and (15), the drag tensors depend only upon the spatial positions of the centers of the spheres and upon their radii. It might appear from Eq. (15) that the solutions for all of the velocity and pressure fields, namely $(\mathbf{U}^{VI},\mathbf{Q}^{VI})$, $(\mathbf{U}^{\Omega I},\mathbf{Q}^{\Omega I})$, $(\mathbf{U}^{SI},\mathbf{Q}^{SI})$ and $(\mathbf{U}^{EI},\mathbf{Q}^{EI})$, are required to calculate the drag coefficients. However, this is not so! Rather, upon utilizing the Lorentz reciprocal theorem it can be shown (Appendix A) that knowledge of the solutions of only the relatively simplest fields, namely the respective translational and rotational fields ($\mathbf{U}^{VI}$,$\mathbf{Q}^{VI}$) and ($\mathbf{U}^{\Omega I}$,$\mathbf{Q}^{\Omega I}$), suffice to determine all of the drag coefficients.

The resistance tensors possess various symmetry properties. Symmetry attributes of these tensors for the respective cases of translational and rotational motions, as well as for homogeneous shear, were addressed by Brenner and O'Neill (1972). Appendix B provides a short recapitulation of these properties, as well as material on the uniqueness and symmetry properties of the new resistance tensors **F** and **T** arising from the existence of quadratic terms in the undisturbed fluid velocity field.

3.2. Two-body system

The geometry of a two-sphere system is characterized by the distance $2H$ between the sphere centers, their respective radii r_1 and r_2, and the direction of a unit vector **e** pointing from the center of particle 1 to that of particle 2. In Eq. (15), it was shown that the resistance tensors depend only upon system configuration and particle size. Thus, these tensors necessarily possess tensorial forms expressible solely in terms of **e**, the isotropic idemtensor **I** and/or the permutation tensor $\boldsymbol{\varepsilon}$, as well as scalar or pseudoscalar multiples thereof depending only upon r_1,r_2 and H, such that the various terms appearing in the explicit representations of these tensors accord with their known symmetry properties and parity (i.e., their true- or pseudo-tensor properties). Hence,

$$K_{ij}^{IK} = k_A^{IK}\delta_{ij} + k_B^{IK}e_ie_j, \quad L_{ij}^{IK} = l_A^{IK}\delta_{ij} + l_B^{IK}e_ie_j, \tag{16a}$$

$$D_{ij}^{IK} = d^{IK}\varepsilon_{ijk}e_k, \quad C_{ij}^{IK} = c^{IK}\varepsilon_{ijk}e_k, \tag{16b}$$

$$\Phi_{ijk}^{IK} = \phi_A^{IK}\left(\delta_{ij}e_k + \delta_{ik}e_j\right) + \phi_B^{IK}e_ie_je_k, \quad \Psi_{ijk}^{IK} = \psi^{IK}(\varepsilon_{ijl}e_le_k + \varepsilon_{ikl}e_le_j), \tag{16c}$$

$$F_{ijkl}^{IK} = f_A^{IK}\delta_{ij}\delta_{kl} + f_B^{IK}\delta_{ij}e_ke_l + f_C^{IK}\delta_{kl}e_ie_j + f_D^{IK}\left(\delta_{ik}e_je_l + \delta_{il}e_je_k\right) + f_E^{IK}e_ie_je_ke_l, \tag{16d}$$

$$T_{ijkl}^{IK} = t_A^{IK}(\varepsilon_{ijk}e_l + \varepsilon_{ijl}e_k) + t_B^{IK}\varepsilon_{ijm}e_m\delta_{kl} + t_C^{IK}\varepsilon_{ijm}e_me_ke_l. \tag{16e}$$

The last two expressions are both new. All the tensorial components (lower case letters) are functions only of H, r_1 and r_2. The k, l, c, d, ϕ and ψ coefficients appearing in Eq. (16) were implicitly or explicitly calculated by earlier investigators, numerically and/or analytically. The f and t coefficients appearing in Eq. (16) are calculated here for the first time (using the method of reflections; see Appendix C). For the sake of completeness, as well as to establish a common notation among the known results of prior investigations, analytical expressions for each of the resistance tensors are provided in Appendix D. Their respective orders of accuracy were chosen such that subsequent calculations of the migration velocities of the freely suspended spheres in quadratic flows (the latter characterized by a length scale R_0) could be consistently obtained up to $O[(r_1/H)^{\alpha}(r_2/H)^{\beta}(r_1/R_0)^{\gamma}(r_2/R_0)^{\delta}]$, in which $\alpha+\beta+\gamma+\delta<6$.

3.2.1. Two spheres freely suspended in quadratic flows

The general solution for two spheres *freely* suspended in quadratic flows can be obtained from Eqs. (14) and (16). Since $\mathbf{f}^I$ and $\mathbf{t}^I$ each vanish when no external forces or couples act upon either of the spheres (the neutrally-buoyant case) the four vector equations (14) can be solved for the 'slip velocity' vectors $\mathbf{U}^I-\mathbf{v}^I$ and $\boldsymbol{\Omega}^I-\boldsymbol{\omega}^I$ $(I=1{,}2)$ in terms of the known values of $\mathbf{S}^I$ and $\mathbf{E}$ characterizing the undisturbed field. The resulting 12 scalar linear equations can be presented in the partitioned matrix (tensor) form:

$$\begin{bmatrix} \mathbf{K}^{11} & \mathbf{D}^{11} & \mathbf{K}^{12} & \mathbf{D}^{12} \\ \mathbf{C}^{11} & \mathbf{L}^{11} & \mathbf{C}^{12} & \mathbf{L}^{12} \\ \mathbf{K}^{21} & \mathbf{D}^{21} & \mathbf{K}^{22} & \mathbf{D}^{22} \\ \mathbf{C}^{21} & \mathbf{L}^{21} & \mathbf{C}^{22} & \mathbf{L}^{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}^{(1)}-\mathbf{v}^{(1)} \\ \boldsymbol{\Omega}^{(1)}-\boldsymbol{\omega}^{(1)} \\ \mathbf{V}^{(2)}-\mathbf{v}^{(2)} \\ \boldsymbol{\Omega}^{(2)}-\boldsymbol{\omega}^{(2)} \end{bmatrix} = -\begin{bmatrix} \boldsymbol{\Phi}^{11}{:}\mathbf{S}^{(1)}+\boldsymbol{\Phi}^{12}{:}\mathbf{S}^{(2)}+\left(\mathbf{F}^{11}+\mathbf{F}^{12}\right)\vdots\mathbf{E} \\ \boldsymbol{\Psi}^{11}{:}\mathbf{S}^{(1)}+\boldsymbol{\Psi}^{12}{:}\mathbf{S}^{(2)}+\left(\mathbf{T}^{11}+\mathbf{T}^{12}\right)\vdots\mathbf{E} \\ \boldsymbol{\Phi}^{21}{:}\mathbf{S}^{(1)}+\boldsymbol{\Phi}^{22}{:}\mathbf{S}^{(2)}+\left(\mathbf{F}^{21}+\mathbf{F}^{22}\right)\vdots\mathbf{E} \\ \boldsymbol{\Psi}^{21}{:}\mathbf{S}^{(1)}+\boldsymbol{\Psi}^{22}{:}\mathbf{S}^{(2)}+\left(\mathbf{T}^{21}+\mathbf{T}^{22}\right)\vdots\mathbf{E} \end{bmatrix}. \tag{17}$$

Eq. (17) is of the form $\mathbf{R}\times\mathbf{X}=\mathbf{Y}$, where $\mathbf{R}$ is the 12×12 LHS grand resistance matrix, $\mathbf{X}$ is the 12×1 generalized slip velocity column vector, and $\mathbf{Y}$ is the 12×1 RHS generalized undisturbed flow column vector.

The grand resistance matrix $\mathbf{R}$ is both positive definite and symmetric (see Appendix B). As such, its inverse exists. The inverse matrix $\mathbf{R}^{-1}$ is a second-rank tensor of 12 dimensions, possessing a representation which depends only upon scalar or pseudoscalar multiples of the basic physical directional quantities entering in the problem, namely the unit vector $\mathbf{e}$, the idemtensor $\mathbf{I}$ and permutation tensor $\boldsymbol{\varepsilon}$. It can easily be shown that its general form is

$$\mathbf{R}^{-1}=\begin{bmatrix} \mathbf{R}^{11} & \mathbf{P}^{12} & \mathbf{R}^{13} & \mathbf{P}^{14} \\ \mathbf{P}^{21} & \mathbf{R}^{22} & \mathbf{P}^{23} & \mathbf{R}^{24} \\ \mathbf{R}^{31} & \mathbf{P}^{32} & \mathbf{R}^{33} & \mathbf{P}^{34} \\ \mathbf{P}^{41} & \mathbf{R}^{42} & \mathbf{P}^{43} & \mathbf{R}^{44} \end{bmatrix}, \tag{18}$$

where $\mathbf{R}^{IJ}$ is a second-rank true tensor and $\mathbf{P}^{IJ}$ a second-rank *pseudo*-tensor. Hence,

$$R_{ij}^{IJ}=\alpha^{IJ}\delta_{ij}+\beta^{IJ}e_ie_j, \qquad P_{ij}^{IJ}=\gamma^{IJ}\varepsilon_{ijk}e_k, \tag{19}$$

where the coefficients α^{IJ},β^{IJ}, and $(I,J=1{,}2{,}3{,}4)$ are constants depending upon r_1,r_2 and H.

Upon post-multiplying $\mathbf{R}^{-1}$ by $\mathbf{Y}$ it follows from Eqs. (16), (18) and (19) that the translational and rotational slip velocities of the spheres, respectively, possess the general forms

$$V_i^I-v_i^I=\tilde{F}_{ijkl}^I E_{jkl}+\sum_{K=1}^{2}\tilde{\Phi}_{ijk}^{IK}S_{jk}^K, \tag{20a}$$

$$\Omega_i^I - \omega_i^I = \tilde{T}_{ijkl}^I E_{jkl} + \sum_{K=1}^{2} \tilde{\Psi}_{ijk}^{IK} S_{jk}^K, \tag{20b}$$

in which

$$\tilde{\Phi}_{ijk}^{IK} = \tilde{\phi}_A^{IK}\left(\delta_{ij}e_k + \delta_{ik}e_j\right) + \tilde{\phi}_B^{IK} e_i e_j e_k, \quad \tilde{\Psi}_{ijk}^{IK} = \tilde{\psi}^{IK}(\varepsilon_{ijl}e_l e_k + \varepsilon_{ikl}e_l e_j), \tag{21a}$$

$$\tilde{F}_{ijkl}^{IK} = \tilde{f}_A^I \delta_{ij}\delta_{kl} + \tilde{f}_B^I \delta_{ij} e_k e_l + \tilde{f}_C^I \delta_{kl} e_i e_j + \tilde{f}_D^I\left(\delta_{ik} e_j e_l + \delta_{il} e_j e_k\right) + \tilde{f}_E^I e_i e_j e_k e_l, \tag{21b}$$

$$\tilde{T}_{ijkl}^I = \tilde{t}_A^I(\varepsilon_{ijk}e_l + \varepsilon_{ijl}e_k) + \tilde{t}_B^I \varepsilon_{ijm} e_m \delta_{kl} + \tilde{t}_C^I \varepsilon_{ijm} e_m e_k e_l. \tag{21c}$$

Lower case symbols appearing beneath the tildes in the above equations are constants (not equal to the constants in Eq. (16)), depending only upon r_1,r_2 and H. Eqs. (20) and (21) constitute general expressions for the motions of the two spheres moving through quadratic flow fields. Obviously, use of Eq. (2c) makes it possible to rewrite Eq. (20) such that the final expression for the slip velocity vector depends only upon a shear rate evaluated at a *single* arbitrary point O rather than upon the respective pair of local shear rates at the centers of the two spheres. The representation given in Eq. (20) is preferable, since it does not distinguish between the centers of the two particles. All of the coefficients appearing in Eq. (21) can be directly calculated from the grand resistance matrix $\mathbf{R}$ and generalized flow vector $\mathbf{Y}$.

Upon applying the method-of-reflections-derived expressions for the grand resistance matrix and the generalized flow vector from Appendix D, the translational slip velocity of sphere 1 is found to be

$$\begin{aligned}\hat{\mathbf{V}}^{(1)} - \hat{\mathbf{v}}^{(1)} = {} & \frac{1}{3}\hat{r}_1^2\hat{\mathbf{E}}{:}\mathbf{I} + \hat{r}_2^2\varepsilon_2^3\left(\frac{5}{192}\hat{\mathbf{E}}{:}\mathbf{I} + \frac{5}{32}\mathbf{ee}{:}\hat{\mathbf{E}} - \frac{3}{64}\hat{\mathbf{E}}{:}\mathbf{ee} + \frac{5}{64}\mathbf{ee}\cdot\hat{\mathbf{E}}{:}\mathbf{I} - \frac{35}{64}\mathbf{eeee}\,\vdots\,\hat{\mathbf{E}}\right) \\ & + \frac{5}{8}\hat{r}_2\varepsilon_2^2\left(1 - \frac{9}{16}\varepsilon_1\varepsilon_2\right)\mathbf{eee}{:}\hat{\mathbf{S}}^{(2)} + \frac{1}{16}\hat{r}_2\varepsilon_2^4\mathbf{e}\cdot\hat{\mathbf{S}}^{(2)} + O(6),\end{aligned} \tag{22}$$

where

$$\varepsilon_1 = r_1/H, \qquad \varepsilon_2 = r_2/H,$$

and in which the symbol $O(N)$ denotes an error of $O(\varepsilon_1^n\varepsilon_2^m\hat{r}_1^p\hat{r}_2^q)$, where $n+m+p+q=N$. Here and henceforth, all careted symbols are dimensionless. Velocities and lengths are respectively scaled with respect to a characteristic velocity V_0 and characteristic length R_0 of the quadratic flow. Thus,

$$\hat{\mathbf{S}} = \mathbf{S}R_0/V_0, \quad \hat{\mathbf{E}} = \mathbf{E}R_0^2/V_0, \quad \hat{r}_1 = r_1/R_0, \quad \hat{r}_2 = r_2/R_0.$$

To avoid possible confusion between superscripts denoting both exponential powers and particle number labeling, parentheses were added to the latter superscripts in Eq. (22). An expression comparable to Eq. (22) may be written down for the velocity of sphere 2 upon interchanging all 1 and 2 superscripts and subscripts while replacing $\mathbf{e}$ by $-\mathbf{e}$, where $\mathbf{e}$ denotes a unit vector pointing from the center of particle 1 to the center of particle 2.

Schonberg et al. (1986) addressed the more limited two-dimensional case of a two-sphere system in a plane Poiseuille flow. He obtained an $O(3)$ solution that is identical to the penultimate term of Eq. (22). Note that although $\mathbf{F}^{IK}$ and $\mathbf{T}^{IK}$ are each only of $O(4)$, this limited accuracy nevertheless suffices in order to obtain $O(6)$ results for the particle velocities after inversion of the grand resistance matrix (albeit only for neutrally-buoyant particles).

The RHS of Eq. (22) determines the slip velocity of particle 1 relative to the undisturbed velocity $\mathbf{v}^{(1)}$. The mean velocity $\mathbf{V}_{\mathrm{CG}}$ of the 'center of gravity' (i.e., center of volume) of the two-body system in the case of two identical spheres is

$$\begin{aligned}\hat{\mathbf{V}}_{\mathrm{CG}} &= \frac{1}{2}\left(\hat{\mathbf{V}}^{(1)}+\hat{\mathbf{V}}^{(2)}\right) = \frac{1}{2}\left(\hat{\mathbf{v}}^{(1)}+\hat{\mathbf{v}}^{(2)}\right)+\frac{1}{3}\hat{r}^2\hat{\mathbf{E}}{:}\mathbf{I}\\ &+\hat{r}^2\varepsilon^3\left(\frac{5}{192}\hat{\mathbf{E}}{:}\mathbf{I}+\frac{5}{32}\mathbf{ee}{:}\hat{\mathbf{E}}-\frac{3}{64}\hat{\mathbf{E}}{:}\mathbf{ee}+\frac{5}{64}\mathbf{ee}\cdot\hat{\mathbf{E}}{:}\mathbf{I}-\frac{35}{64}\mathbf{eeee}\vdots\hat{\mathbf{E}}\right)\\ &+\frac{5}{16}\hat{r}\varepsilon^2\left(1-\frac{9}{16}\varepsilon_2\right)\mathbf{eee}{:}\left(\hat{\mathbf{S}}^{(2)}-\hat{\mathbf{S}}^{(1)}\right)+\frac{1}{32}\hat{r}\varepsilon^4\mathbf{e}\cdot\left(\hat{\mathbf{S}}^{(2)}-\hat{\mathbf{S}}^{(1)}\right)+O(6).\end{aligned} \tag{23}$$

In effect, the latter equation furnishes the instantaneous velocity of the midpoint of the line segment joining the sphere centers. For particles of equal size and density this velocity can be viewed as the center-of-mass velocity of the system. Obviously, the lateral migration velocity vanishes for a monodisperse suspension immersed in a simple shear flow, for which circumstances $r=r_1=r_2, \varepsilon=\varepsilon_1=\varepsilon_2, \mathbf{S}^{(1)}=\mathbf{S}^{(2)}$, and $\mathbf{E}=\mathbf{0}$. However, in quadratic flows, $\mathbf{E}$ is nonzero and $\mathbf{S}^{(1)}\neq\mathbf{S}^{(2)}$. Hence, in general, *instantaneous lateral migration across the undisturbed streamlines exists for two spheres freely suspended in a quadratic flow*. Subsequently, we will show that this *instantaneous* migration is, in fact, *permanent* in time for a two-sphere system immersed in an axisymmetric Poiseuille flow.

The last two terms in Eq. (23) could have been rewritten exclusively in terms of $\mathbf{E}$ and $\mathbf{e}$ by using Eq. (2c). However, we prefer to retain Eq. (23) in its present form, since it provides better physical insight into the source of the penultimate term in Eq. (23), and — as discussed in the next section — the latter contribution proves to be most significant.

3.2.2. Two spheres freely suspended in an axisymmetric Poiseuille flow

Consider a system consisting of two neutrally-buoyant spheres suspended in an *unbounded* Poiseuille flow. Introduce a circular cylindrical coordinate system (R,ϕ,z) whose origin lies along the tube's axis of symmetry, and define the corresponding set of unit vectors $(\mathbf{i}_R,\mathbf{i}_\phi,\mathbf{i}_z)$. The dimensionless velocity field, shear rate, and shear-rate gradient are, respectively,

$$\hat{\mathbf{v}}^{\infty}=\left(1-\hat{R}^2\right)\mathbf{i}_z,\quad \hat{\mathbf{S}}=-\hat{R}(\mathbf{i}_R\mathbf{i}_z+\mathbf{i}_z\mathbf{i}_R),\quad \hat{\mathbf{E}}=-\mathbf{i}_z(\mathbf{I}-\mathbf{i}_z\mathbf{i}_z). \tag{24}$$

Here, the centerline velocity V_0 was used to scale velocities, whereas the radius R_0 (where the local fluid velocity is zero) was used to scale lengths. With the centers of the spheres respectively located at (R_1,ϕ_1,z_1) and (R_2,ϕ_2,z_2), and with $\mathbf{i}_{R_1}$ and $\mathbf{i}_{R_2}$ denoting the respective unit radial vectors pointing towards the particle centers, we have that

$$\hat{\mathbf{S}}^{(1)} = -\hat{R}_1\left(\mathbf{i}_{R_1}\mathbf{i}_z + \mathbf{i}_z\mathbf{i}_{R_1}\right), \quad \hat{\mathbf{S}}^{(2)} = -\hat{R}_2\left(\mathbf{i}_{R_2}\mathbf{i}_z + \mathbf{i}_z\mathbf{i}_{R_2}\right). \tag{25}$$

Interest focuses on the radial migration, if any, of the two-body system. In this context we note that the radial component of the center-of-gravity velocity is given by

$$\hat{V}_{R_{\mathrm{CG}}} = \frac{\hat{\mathbf{V}}_{\mathrm{CG}} \cdot \left(\hat{R}_1\mathbf{i}_{R_1} + \hat{R}_2\mathbf{i}_{R_2}\right)}{\hat{R}_{\mathrm{CG}}}, \tag{26}$$

where

$$\hat{R}_{\mathrm{CG}} = \left[\hat{R}_2^2 + \hat{R}_1^2 + 2\hat{R}_1\hat{R}_2\cos\left(\phi_2 - \phi_1\right)\right]^{1/2}. \tag{27}$$

Substitution of Eqs. (24)–(26) into Eq. (23) yields

$$\hat{V}_{R_{\mathrm{CG}}} = \frac{(\mathbf{i}_z \cdot \mathbf{e})}{\hat{R}_{\mathrm{CG}}}\left\{\frac{5}{8}\hat{r}\varepsilon^2\left(1 - \frac{9}{16}\varepsilon^2\right)\left[\hat{R}_2^2\left(\mathbf{i}_{R_2} \cdot \mathbf{e}\right)^2 - \hat{R}_1^2\left(\mathbf{i}_{R_1} \cdot \mathbf{e}\right)^2\right] + \frac{1}{32}\hat{r}\varepsilon^4\left(\hat{R}_2^2 - \hat{R}_1^2\right)\right.$$
$$\left. - \frac{15}{64}\hat{r}^2\varepsilon^3\left[1 + \frac{7}{3}(\mathbf{i}_z \cdot \mathbf{e})^2\right]\left[\hat{R}_1\left(\mathbf{i}_{R_1} \cdot \mathbf{e}\right) + \hat{R}_2\left(\mathbf{i}_{R_2} \cdot \mathbf{e}\right)\right] + O(6)\right\}. \tag{28}$$

Obviously, Eq. (28) must remain unaltered if suffices 1 and 2 are interchanged (in which case **e** must also be replaced by −**e**). Now,

$$\mathbf{e} = \frac{\hat{R}_2\mathbf{i}_{R_2} - \hat{R}_1\mathbf{i}_{R_1} + \mathbf{i}_z(\hat{z}_2 - \hat{z}_1)}{2\hat{H}}. \tag{29}$$

Hence,

$$\hat{R}_2^2\left(\mathbf{i}_{R_2} \cdot \mathbf{e}\right)^2 - \hat{R}_1^2\left(\mathbf{i}_{R_1} \cdot \mathbf{e}\right)^2 = \left(\hat{R}_2^2 - \hat{R}_1^2\right)\frac{\left[\hat{R}_2^2 + \hat{R}_1^2 - 2\hat{R}_1\hat{R}_2\cos\left(\phi_2 - \phi_1\right)\right]}{4\hat{H}^2}, \tag{30a}$$

$$\hat{R}_2\left(\mathbf{i}_{R_2} \cdot \mathbf{e}\right) + \hat{R}_1\left(\mathbf{i}_{R_1} \cdot \mathbf{e}\right) = \frac{\left(\hat{R}_2^2 - \hat{R}_1^2\right)}{2\hat{H}}, \tag{30b}$$

where $2\hat{H}$ is the dimensionless center-to-center distance between spheres, namely

$$2\hat{H} = \left[\hat{R}_2^2 + \hat{R}_1^2 - 2\hat{R}_1\hat{R}_2\cos\left(\phi_1 - \phi_2\right) + (\hat{z}_2 - \hat{z}_1)^2\right]^{1/2}. \tag{31}$$

Substitution of Eqs. (29) and (30) into Eq. (28) further simplifies the expression for the radial velocity, yielding

$$V_{R_{\mathrm{CG}}} = -\left(\hat{R}_2^2 - \hat{R}_1^2\right)(\hat{z}_2 - \hat{z}_1)\left\{\frac{5}{64}\hat{r}\varepsilon^2\left(1 - \frac{9}{16}\varepsilon^2\right)\frac{\left[\hat{R}_2^2 + \hat{R}_1^2 - 2\hat{R}_1\hat{R}_2\cos(\phi_1 - \phi_2)\right]}{\hat{H}^3\hat{R}_{\mathrm{CG}}}\right.$$
$$\left. + \frac{1}{64}\hat{r}\varepsilon^4\frac{1}{\hat{H}\hat{R}_{\mathrm{CG}}} - \frac{15}{256}\hat{r}^2\varepsilon^3\left[1 - \frac{7}{12}\frac{(\hat{z}_2 - \hat{z}_1)^2}{\hat{H}^2}\right]\frac{1}{\hat{H}^2\hat{R}_{\mathrm{CG}}}\right\}. \quad (32)$$

4. Conclusions

Several important conclusions may be drawn from Eq. (32). The expression in curly parentheses is always positive. As such, the direction of lateral migration will be determined by the *relative* radial and axial locations of the spheres, but not by their relative circumferential location. The latter polar angle difference affects only the *magnitude* of the radial velocity. It is also clear from Eq. (32) that the magnitude and direction of the radial velocity is indifferent to particle numbering. For the special case of two spheres located at equal radial distances ($R_2 - R_1 = 0, z_2 - z_1 \neq 0$) no radial migration is predicted. However, in the case where $z_2 - z_1 \neq 0$ and $\phi_1 - \phi_2 = \pi$ (corresponding to two particles placed on either side of the centerline with one situated ahead of the other), R_{CG} vanishes. As a result, both the numerator and denominator of Eq. (32) vanish. In this limit, radial migration exists in a direction away from the centerline, where the leading particle migrates towards the centerline whereas the trailing particle moves away from the centerline. (This fact can be deduced simply from Eq. (22).) This result was also obtained by Schonberg et al. (1986) for the case of a two-particle system suspended in a plane Poiseuille flow. The case of particles located at equal axial positions ($z_2 - z_1 = 0$) is much less important, since particles occupying different *radial* positions will experience different *axial* velocities and, hence, will soon thereafter acquire different axial positions.

Consider two generic cases:

(A) Particle 1 is located *closer* to the flow symmetry axis ($R_1 < R_2$), and is axially *leading* particle 2 ($z_1 > z_2$);
(B) Particle 1 is located *closer* to the flow symmetry axis ($R_1 < R_2$) and is axially *trailing* particle 2 ($z_1 < z_2$).

In case A, the radial migration is positive and the system tends to migrate away from the symmetry axis of the flow. Notice that this configuration will persist since particle 1 possesses a higher axial velocity. Thus, *the system moves radially from regions of low shear rates to higher ones*. This result contrasts with the commonly accepted notion that particles always tend to migrate from regions of high to low shear rate (and consequently that a positive particle-flux diffusion coefficient can be defined to account for the role of shear-rate gradients in effecting the transport of non-Brownian particles in concentrated suspensions). Such 'reverse migration' was observed experimentally by Shauly et al. (1997), who wrote that "particles can migrate against (emphasis ours) concentration gradients." We suggest that the mechanism that alters the migration direction is caused by the shear-rate gradients explored here.

Superficially, it seems that the foregoing important conclusions are reversed in case B where the inner particle 1 lags behind particle 2. However, this configuration would last for only a brief period of time, since the inner particle possesses a higher axial velocity and hence would soon pass the outer, slower-moving particle. Thus, the configuration of case A will eventually be recovered, whereupon the circumstances governing that case would, henceforth, govern the subsequent behavior of the system.

The most significant term in Eq. (32) contributing to the migration behavior, described above, arises from the difference in shear rates existing at the centers of the spheres (which is reflected in the penultimate term in Eq. (32)). We suggest without proof that this would also likely occur for flows of higher order than quadratic, in which case the contention that lateral migration occurs from low to high shear rate regions would not be restricted to quadratic flows. The magnitude of this term suggests that the effect is linearly proportional to both the volumetric particle concentration and to the variation in the undisturbed shear rate occurring on a length scale corresponding to the distance between sphere centers.

In summary, this paper has addressed the contribution made by the quadratic term appearing in the undisturbed velocity field to the drag forces and torques exerted on the individual spheres in a dilute, albeit hydrodynamically interacting, multiparticle system. The method of reflections was applied (Appendix C) to obtain asymptotic expressions for the drag forces and torques exerted on each of the spheres in a two-sphere system immersed in a general quadratic flow in terms of their center-to-center spacing and the orientation of their line-of-centers relative to the principal axes of the undisturbed shear. Issues of the uniqueness and symmetry properties of the new intrinsic Stokes resistance coefficients thereby calculated were analyzed in Appendix B. Results obtained from the general solution were employed to derive the respective velocities of two neutrally- buoyant spheres freely suspended in unbounded quadratic flows. The additional drag force arising from the quadratic terms, above and beyond the contributions arising for *linear* shear flows, was shown to cause cross-streamline motion of the center of gravity of the two-sphere system. The particular case of two identical, freely suspended, hydrodynamically-interacting spheres in an axisymmetric Poiseuille flow was deduced from these more general results. It was shown in this case that the center of gravity of the two-body system migrates laterally from regions of low to high shear rates.

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Appendix A.

The Lorentz reciprocal theorem (Happel and Brenner, 1983) for Stokes flows states for a given instantaneous geometric configuration of a hydrodynamic system that two different fluid fields, say, $(\mathbf{v}',p',\boldsymbol{\pi}')$ and $(\mathbf{v}'',p'',\boldsymbol{\pi}'')$, arising from two different velocity or stress fields prevailing at the particle surfaces and/or external boundaries, satisfy the following integral condition over the boundaries:

$$\int_{\partial S} v_i' \pi_{in}'' n_n \, \mathrm{d}s = \int_{\partial S} v_i'' \pi_{in}' n_n \, \mathrm{d}S, \tag{A1}$$

where $(\mathbf{v},p,\boldsymbol{\pi})$ are the respective velocity, pressure, and stress fields, $\mathbf{n}$ is a unit normal vector, and ∂S connotes all the internal or external boundaries of the fluid.

Denote by $(\mathbf{v}',p',\boldsymbol{\pi}')$ the trio of fields generated by the translational motion $\mathbf{V}^I$ of particle I suspended in an otherwise quiescent flow field (i.e., an undisturbed velocity field which vanishes at infinity) and where simultaneously all the other particles are at rest. Explicitly, this trio of fields is of the form $(U_{ij}^{VI},Q_j^{VI},\Pi_{nij}^{VI})V_j^I$, where the dyadic velocity field U_{ij}^{VI} satisfies boundary conditions (11a). In addition, denote by $(\mathbf{v}'',p'',\boldsymbol{\pi}'')$ the trio of fields generated by the action of a homogeneous shear-rate gradient $\mathbf{E}$ over particle J with all the particles at rest; explicitly, this latter trio of fields is of the form $(U_{ijkl}^{EJ},Q_{jkl}^{EJ},\Pi_{nijkl}^{EJ})E_{jkl}$, where the tetradic velocity field U_{ijkl}^{EJ} satisfies boundary conditions (11d), with J replacing I. Since $\mathbf{V}^I$ and $\mathbf{E}$ are position-independent, arbitrary tensors, introduction of these velocity and stress fields into Eq. (A1) yields

$$\int_{\Sigma_I \partial S^I} U_{ij}^{VI} \Pi_{niklm}^{EJ} n_n \, \mathrm{d}S = \int_{\Sigma_I \partial S^I} U_{iklm}^{EJ} \Pi_{nij}^{VI} n_n \, \mathrm{d}S \tag{A2}$$

for any $I,J = 1,2,\ldots,N$.Substitution of boundary conditions (11a), (11d) into Eq. (A2) yields

$$-\int_{\partial S^I} \Pi_{njklm}^{EJ} n_n \, \mathrm{d}S = \int_{\partial S^J} x_m^J x_l^J \Pi_{nkj}^{VI} n_n \, \mathrm{d}S. \tag{A3}$$

However, the LHS of Eq. (A3) is equal to the drag resistance tensor F_{jklm}^{IJ} defined in Eq. (15). Hence,

$$F_{jklm}^{IJ} = \int_{\partial S^J} x_m^J x_l^J \Pi_{nkj}^{VI} n_n \, \mathrm{d}S. \tag{A4}$$

Thus, the drag coefficient arising from the quadratic portion of the undisturbed flow can easily be obtained from knowledge of the triadic stress field Π_{nkj}^{VI} by an elementary quadrature. Determination of the latter stress field requires solving a very much simpler boundary-value problem than would otherwise appear necessary, had one actually to solve the original quadratic field problem to obtain the stress field Π_{nijkl}^{EK} seemingly required by Eq. (15) to compute the tensor F_{ijkl}^{IK} via the defining formula

$$F_{ijkl}^{IK} = -\int_{\delta S^I} \Pi_{nijkl}^{EK} n_n \, \mathrm{d}S.$$

Explicitly, modulo the quadrature required in Eq. (A4), to calculate $\mathbf{F}$ one need only to solve the 'elementary' boundary-value problem arising from the translational motion of a single particle in the multiparticle system in an otherwise quiescent fluid within which all the other particles remain at rest. Analogously,

$$T^{IJ}_{jklm} = \int_{\partial S^J} x^J_m x^J_l \Pi^{\Omega I}_{nkj} n_n \, \mathrm{d}S. \tag{A5}$$

A similar approach requiring explicit knowledge only of the field generated by the *rotation* of the particle of interest in a fluid (where all other particles as well as the fluid at infinity are at rest) can be utilized to obtain **T** in Eq. (A5). In this manner, all the resistance tensors defined generically in Eq. (15) can be calculated.

Appendix B.

The Stokes resistance tensors defined in Eq. (15) satisfy an array of symmetry-related properties, most of which can easily be obtained by a simple application of the Lorentz reciprocal theorem. Resistance tensors for translational motions, rotational motions, and homogeneous shear flows have systematically been investigated in the past, to the extent that those tensors describing these classes of motion may be regarded as known for all possible combinations of sphere sizes and separation distances. Accordingly, we shall recapitulate only those properties pertinent to our analysis. Uniqueness and symmetry (both kinetic and geometric) properties of the resistance tensors **F** and **T** for the shear-rate gradient case are also briefly discussed in what follows.

The following kinetic symmetry relations apply to the grand resistance matrix **R**:

$$K^{IJ}_{ij} = K^{JI}_{ji}, \quad D^{IJ}_{ij} = C^{JI}_{ji}, \quad L^{IJ}_{ij} = L^{JI}_{ji}. \tag{B1}$$

Due to the symmetry of the shear-rate tensor we also have that

$$\Phi^{IJ}_{ijk} = \Phi^{IJ}_{ikj}, \quad \Psi^{IJ}_{ijk} = \Psi^{IJ}_{ikj}. \tag{B2}$$

Symmetry of the triadic **E** with respect to its second and third tensor indices results in the following symmetry properties of **F** and **T**:

$$F^{IJ}_{ijkl} = F^{IJ}_{ijlk}, \quad T^{IJ}_{ijkl} = T^{IJ}_{ijlk}. \tag{B3}$$

From the divergence-free condition $E_{iij} = 0$ imposed by fluid incompressibility upon **E**, we find that the tensor F^{IJ}_{ijkl} is unique only up to an additive fourth-rank tensor of the form $A_{il}\delta_{jk}$, where A_{ij} is an arbitrary second-rank tensor. A similar property exists for **T**. Thus, $F^{IJ}_{ijkl} + A_{il}\delta_{jk}$ and $T^{IJ}_{ijkl} + B_{il}\delta_{jk}$ constitute equally valid representations of the respective resistance tensors. The drag force and torque exerted on a particle are independent of the respective choices of the arbitrary tensors **A** and **B**.

Appendix C.

Consider two spherical particles of respective radii r_1 and r_2 whose centers are separated by a distance $2H$, and which are suspended in a fluid of viscosity μ. The velocity and pressure fields ($\mathbf{u}$,μq) are governed by Stokes equations. The velocity field satisfies the following conditions over the particles boundaries (see Eq. (11d)):

$$u_i = \left\{ \begin{array}{ll} -E_{ijk}x_jx_k & \text{on } \partial S^1, \\ 0 & \text{on } \partial S^2, \end{array} \right\} \tag{C1}$$

where $\mathbf{x}$ is measured from the center of sphere 1.

The general solutions for the velocity, pressure, and stress fields (see Eqs. (9a)–(9c)) are, respectively, of the forms

$$u_i = U_{ijkl}E_{jkl}, \quad q = Q_{jkl}E_{jkl}, \quad \pi_{ni} = \Pi_{nijkl}E_{jkl}. \tag{C2}$$

Here, the superscript $E1$ has been suppressed from the tensorial field symbols in order to simplify the appearance of the expressions which follow. From Eqs. (C1) and (C2),

$$U_{ijkl} = \left\{ \begin{array}{ll} -\delta_{ij}x_lx_k & \text{on } \partial S^1, \\ 0 & \text{on } \partial S^2. \end{array} \right\} \tag{C3}$$

The fields ($\mathbf{U}$, $\mathbf{Q}$, $\boldsymbol{\Pi}$) may be obtained by using the method of reflections (Happel and Brenner, 1983):

$$U_{ijkl} = U^{(1)}_{ijkl} + U^{(2)}_{ijkl} + U^{(3)}_{ijkl} + \cdots, \tag{C4a}$$

$$Q_{jkl} = Q^{(1)}_{jkl} + Q^{(2)}_{jkl} + Q^{(3)}_{jkl} + \cdots, \tag{C4b}$$

$$\Pi_{nijkl} = \Pi^{(1)}_{nijkl} + \Pi^{(2)}_{nijkl} + \Pi^{(3)}_{nijkl} + \cdots, \tag{C4c}$$

where fields with odd-numbered superscripts satisfy the requisite conditions imposed at the boundary of particle 1 (but not those at the boundary of particle 2), whereas fields with even-numbered superscripts satisfy the boundary conditions on the surface of particle 2, but not 1. Explicitly,

$$U^{(1)}_{ijkl} = -\delta_{ij}x_kx_l \quad \text{on } \partial S^1, \tag{C5a}$$

$$U^{(2)}_{ijkl} = -U^{(1)}_{ijkl} \quad \text{on } \partial S^2, \tag{C5b}$$

$$U^{(3)}_{ijkl} = -U^{(2)}_{ijkl} \quad \text{on } \partial S^1,\text{etc.} \tag{C5c}$$

The solution for the first reflection is easily found to be

$$V^{(1)}_{ijkl} = -r_1^5\left(r^2 - r_1^2\right)H^{(4)}_{ijkl} - \frac{1}{6}r_1^3\left(r^2 - r_1^2\right)H^{(2)}_{ij}\delta_{kl} - \frac{2}{3}r_1^5H^{(2)}_{kl}\delta_{ij} - \frac{1}{3}r_1^3\delta_{ij}\delta_{kl}H^{(0)}, \tag{C6a}$$

$$P^{(1)}_{jkl} = -\frac{7}{2}r_1^5H^{(3)}_{jkl} - \frac{1}{2}r_1^3\delta_{kl}H^{(1)}_j, \tag{C6b}$$

where

$$H^{(n)}_{i_1 i_2 \dots i_n} = \frac{(-1)^n}{n!}\frac{\partial}{\partial x_{i_1}}\frac{\partial}{\partial x_{i_2}}\cdots\frac{\partial}{\partial x_{i_n}}\left(\frac{1}{r}\right), \qquad H^{(0)} = \frac{1}{r}, \tag{C7}$$

in which $r = |\mathbf{x}|$. An explicit expression for the first reflection is

$$u_i^{(1)} = -\frac{1}{8}r_1^5\left(r^2 - r_1^2\right)\left[\frac{35}{r^9}x_i x_j x_k x_l E_{jkl} - \frac{5}{r^7}\left(x_i x_j E_{jkk} + 2x_k x_j E_{jki} + x_k x_l E_{ikl}\right) + \frac{1}{r^5}E_{ijj}\right]$$
$$+\left[\frac{1}{12}r_1^3\left(r^2 - r_1^2\right)\left(\frac{3x_i x_j}{r^2} - \delta_{ij}\right)\delta_{kl} - \frac{1}{3}r_1^5\left(\frac{3x_l x_k}{r^2} - \delta_{kl}\right)\delta_{ij} - \frac{1}{3r}r_1^3\delta_{ij}\delta_{kl}\right]E_{jkl}, \tag{C8a}$$

$$q^{(1)} = \left\{-\frac{7}{4}r_1^5\left[\frac{5}{r^7}x_j x_k x_l - \frac{1}{r^5}\left(x_j\delta_{kl} + x_l\delta_{jk} + x_k\delta_{jl}\right)\right] - \frac{1}{2}r_1^3\delta_{kl}\frac{x_j}{r^3}\right\}E_{jkl}. \tag{C8b}$$

The drag force and torque exerted on particle 1 due to reflection 1 can easily be calculated as (see Happel and Brenner, 1983)

$$\frac{1}{\mu}f_i^{1E(1)} = 2\pi r_1^3 E_{ijj}, \quad \frac{1}{\mu}t_i^{1E(1)} = 0. \tag{C9}$$

In lieu of calculating the second reflection, we use Faxen's laws to calculate the drag force and torque exerted on particle 2 induced by the first reflection:

$$\mathbf{f}^{2E(2)} = 6\pi\mu r_2\left[\mathbf{u}^{(1)}\right]_{\mathbf{x}=2H\mathbf{e}} + \pi\mu r_2^3\left[\nabla q^{(1)}\right]_{\mathbf{x}=2H\mathbf{e}}, \tag{C10a}$$

$$\mathbf{t}^{2E(2)} = 4\pi\mu r_2^3\left[\nabla\times\mathbf{u}^{(1)}\right]_{\mathbf{x}=2H\mathbf{e}}. \tag{C10b}$$

Substitution of Eq. (C8) into Eqs. (C10a) gives

$$-\frac{1}{\mu}f_i^{2E(2)} = \pi r_2 r_1^2\left\{\frac{3}{4}\varepsilon_1\left(\delta_{ij} + e_i e_j\right)\delta_{kl}E_{jkl}\right.$$
$$-6\varepsilon_1^3\left[\left(\frac{7}{64}E_{jkk}e_j - \frac{35}{64}E_{jkl}e_l e_j e_k\right)e_i + \frac{5}{32}E_{jki}e_j e_k - \frac{3}{64}E_{ijk}e_j e_k + \frac{1}{64}E_{ijj}\right] \tag{C11}$$
$$\left. + O\left(\varepsilon_1^5\right)\right\} + \frac{1}{16}\pi r_2^3\left[\varepsilon_1^3\left(E_{ikk} - 3E_{jkk}e_i e_j\right) + O\left(\varepsilon_1^5\right)\right].$$

The third reflection need not be calculated explicitly if the drag force and torque exerted on particle 1 are to be calculated only to terms of $O(5)$. Indeed, the field $\mathbf{u}^{(2)}$ can be determined to the requisite order of accuracy via the force obtained in Eq. (C11). We may view this force as the strength of the Stokeslet inducing the velocity field:

$$u_i^{(2)} = -\frac{1}{8\pi\mu} f_j^{2E(2)} \left\{ \delta_{ij} + \frac{x_i^{(2)} x_j^{(2)}}{[r^{(2)}]^2} \right\}, \tag{C12}$$

where the radius vector $\mathbf{r}^{(2)}$ is measured from the center of particle 2. The RHS of Eq. (C11) can be rewritten as $F_{ijkl}^{21} E_{jkl}$. Thus, the drag coefficient F_{ijkl}^{21} pertaining to particle 2 due to a quadratic flow boundary condition over particle 1 can be identified. The result of the calculation appears in Appendix D.

Again using Faxen's laws, the drag force and torque induced by $\mathbf{u}^{(2)}$ on particle 1 are, respectively, found to be

$$\mathbf{f}^{1E(3)} = 6\pi\mu r_1 \left[\mathbf{u}^{(2)}\right]_{\mathbf{x}^{(2)}=-2H\mathbf{e}} + \pi\mu r_1^3 \left[\nabla q^{(1)}\right]_{\mathbf{x}^{(2)}=-2H\mathbf{e}}, \tag{C13a}$$

$$\mathbf{t}^{1E(3)} = 4\pi\mu r_1^3 \left[\nabla \times \mathbf{u}^{(2)}\right]_{\mathbf{x}^{(2)}=-2H\mathbf{e}}. \tag{C13b}$$

Upon substituting Eq. (C12) into Eq. (C13a) we obtain

$$-\frac{1}{\mu} f_i^{1E(3)} = -\frac{9}{32}\pi r_2 r_1^2 \left[\varepsilon_1^2 \left(\delta_{ij} + 3e_i e_j\right) E_{jk} + O(4)\right]. \tag{C14}$$

Adding the contributions of Eqs. (C9) and (C14) yields the total drag force exerted on particle 1. Upon rewriting the sum in the form $F_{ijkl}^{11} E_{jkl}$, it is easy to identify that portion of the drag coefficient pertaining to particle 1 arising from the quadratic flow boundary condition on particle 1. The result appears in Appendix D.

Similarly, the torque coefficients T_{ijkl}^{11} and T_{ijkl}^{21} are derived using Eqs. (C10b) and (C13b). They too appear in Appendix D. The terms F_{ijkl}^{22},F_{ijkl}^{12},T_{ijkl}^{22} and T_{ijkl}^{12} are simply obtained by interchanging ε_1 with ε_2 and r_1 with r_2.

Appendix D.

Write $\varepsilon_1 = r_1/H$ and $\varepsilon_2 = r_2/H$, where r_1 and r_2 are the respective sphere radii, $2H$ is the distance between the sphere centers, and e_i is the unit vector directed from the center of sphere 1 to the center of sphere 2. Denote by the symbol $O(N)$ in what follows an error of $O(\varepsilon_1^{n_1}\varepsilon_2^{n_2})$ where $n_1 + n_2 = N$. The drag resistance tensors of the two-sphere system are then given by

$$K_{ij}^{11} = 6\pi r_1 \left[\delta_{ij} + \frac{9}{64}\varepsilon_1\varepsilon_2\left(\delta_{ij} + 3e_i e_j\right) + \frac{3}{64}\varepsilon_1\varepsilon_2\left(-2\varepsilon_1^2 + \frac{27}{4}\varepsilon_1\varepsilon_2 + 3\varepsilon_2^2\right) e_i e_j \right.$$
$$\left. + \frac{3}{128}\varepsilon_1\varepsilon_2\left(\varepsilon_1^2 + \frac{27}{32}\varepsilon_1\varepsilon_2 + 3\varepsilon_2^2\right)\left(\delta_{ij} - e_i e_j\right) + O(6) \right],$$

$$K_{ij}^{22} = 6\pi r_2\left[\delta_{ij} + \frac{9}{64}\varepsilon_1\varepsilon_2\left(\delta_{ij} + 3e_ie_j\right) + \frac{3}{64}\varepsilon_1\varepsilon_2\left(-2\varepsilon_2^2 + \frac{27}{4}\varepsilon_1\varepsilon_2 + 3\varepsilon_1^2\right)e_ie_j\right.$$

$$\left. + \frac{3}{128}\varepsilon_1\varepsilon_2\left(\varepsilon_2^2 + \frac{27}{32}\varepsilon_1\varepsilon_2 + 3\varepsilon_1^2\right)\left(\delta_{ij} - e_ie_j\right) + O(6)\right],$$

$$K_{ij}^{12} = -\frac{9}{4}\pi r_1\left[\varepsilon_2\left(\delta_{ij} + e_ie_j\right) - \frac{\varepsilon_2}{6}\left(\varepsilon_1^2 - \frac{27}{4}\varepsilon_1\varepsilon_2 + \varepsilon_2^2\right)e_ie_j + \frac{\varepsilon_2}{12}\left(\varepsilon_1^2 + \frac{27}{16}\varepsilon_1\varepsilon_2 + \varepsilon_2^2\right)\left(\delta_{ij} - e_ie_j\right)\right.$$

$$\left. + O(5)\right],$$

$$K_{ij}^{21} = K_{ij}^{12};$$

$$L_{ij}^{11} = 8\pi r_1^3\left[\delta_{ij} + \frac{3}{64}\varepsilon_2\varepsilon_1^3\left(\delta_{ij} - e_ie_j\right) + O(6)\right],$$

$$L_{ij}^{22} = 8\pi r_2^3\left[\delta_{ij} + \frac{3}{64}\varepsilon_2\varepsilon_1^3\left(\delta_{ij} - e_ie_j\right) + O(6)\right],$$

$$L_{ij}^{12} = 0.5\pi r_1^3\left[\varepsilon_2^3\left(\delta_{ij} - e_ie_j\right) + O(5)\right],$$

$$L_{ij}^{21} = L_{ij}^{12};$$

$$C_{ij}^{11} = D_{ji}^{11} = -\frac{9}{16}\pi r_1 r_2\left[\varepsilon_1^3\varepsilon_{ijl}e_l + O(5)\right],$$

$$C_{ij}^{22} = D_{ji}^{22} = +\frac{9}{16}\pi r_1 r_2\left[\varepsilon_2^3\varepsilon_{ijl}e_l + O(5)\right],$$

$$C_{ij}^{12} = D_{ji}^{21} = +\frac{3}{2}\pi r_1 r_2\left[\varepsilon_1^2\varepsilon_{ijl}e_l + O(4)\right],$$

$$C_{ij}^{21} = D_{ji}^{12} = -\frac{3}{2}\pi r_1 r_2\left[\varepsilon_2^2\varepsilon_{ijl}e_l + O(4)\right];$$

$$\Phi_{ijk}^{11} = -\frac{9}{4}\pi r_1 r_2\left[\frac{5}{4}\varepsilon_1^3 e_ie_je_k + \frac{1}{32}\varepsilon_1^5\left(\delta_{ij}e_k + \delta_{ik}e_j\right) + O(7)\right],$$

$$\Phi^{22}_{ijk} = +\frac{9}{4}\pi r_1 r_2\left[\frac{5}{4}\varepsilon_2^3 e_i e_j e_k + \frac{1}{32}\varepsilon_2^5\left(\delta_{ij}e_k + \delta_{ik}e_j\right) + O(7)\right],$$

$$\Phi^{12}_{ijk} = -6\pi r_1 r_2\left[\frac{5}{8}\varepsilon_2^2 e_i e_j e_k + \frac{1}{32}\varepsilon_2^4\left(\delta_{ij}e_k + \delta_{ik}e_j\right) + O(6)\right],$$

$$\Phi^{21}_{ijk} = +6\pi r_1 r_2\left[\frac{5}{8}\varepsilon_1^2 e_i e_j e_k + \frac{1}{32}\varepsilon_1^4\left(\delta_{ij}e_k + \delta_{ik}e_j\right) + O(6)\right];$$

$$\Psi^{11}_{ijk} = \frac{3}{64}\pi r_1^2 r_2\left[\varepsilon_1^6(\varepsilon_{ijl}e_l e_k + \varepsilon_{ikl}e_l e_j) + O(8)\right],$$

$$\Psi^{22}_{ijk} = \frac{3}{64}\pi r_2^2 r_1\left[\varepsilon_2^6(\varepsilon_{ijl}e_l e_k + \varepsilon_{ikl}e_l e_j) + O(8)\right],$$

$$\Psi^{12}_{ijk} = \frac{5}{4}\pi r_1^3\left[\varepsilon_2^3(\varepsilon_{ijl}e_l e_k + \varepsilon_{ikl}e_l e_j) + O(5)\right],$$

$$\Psi^{21}_{ijk} = \frac{5}{4}\pi r_2^3\left[\varepsilon_1^3(\varepsilon_{ijl}e_l e_k + \varepsilon_{ikl}e_l e_j) + O(5)\right];$$

$$F^{11}_{ijmn} = -2\pi r_1^3\left\{\left[\delta_{ij} + \frac{9}{64}\varepsilon_1\varepsilon_2\left(\delta_{ij} + 3e_i e_j\right)\right]\delta_{mn} + O(4)\right\},$$

$$F^{22}_{ijmn} = -2\pi r_2^3\left\{\left[\delta_{ij} + \frac{9}{64}\varepsilon_1\varepsilon_2\left(\delta_{ij} + 3e_i e_j\right)\right]\delta_{mn} + O(4)\right\},$$

$$\begin{aligned} F^{12}_{ijmn} = \pi r_2^3\Bigg[&\frac{3}{4}\varepsilon_1\left(\delta_{ij} + e_i e_j\right)\delta_{mn} + \varepsilon_1\varepsilon_2^2\left(\frac{105}{32}e_i e_j e_m e_n - \frac{21}{32}e_i e_j\delta_{mn} - \frac{15}{32}\delta_{in}e_j e_m - \frac{15}{32}\delta_{im}e_j e_n\right. \\ &\left. + \frac{9}{32}\delta_{ij}e_n e_m - \frac{3}{32}\delta_{ij}\delta_{mn}\right) + \frac{1}{16}\varepsilon_1^3\left(\delta_{ij} - 3e_i e_j\right)\delta_{nm} + \frac{27}{256}\varepsilon_2\varepsilon_1^2\left(\delta_{ij} + 7e_i e_j\right)\delta_{nm} + O(5)\Bigg], \end{aligned}$$

$$\begin{aligned} F^{21}_{ijmn} = \pi r_1^3\Bigg[&\frac{3}{4}\varepsilon_2\left(\delta_{ij} + e_i e_j\right)\delta_{mn} + \varepsilon_2\varepsilon_1^2\left(\frac{105}{32}e_i e_j e_m e_n - \frac{21}{32}e_i e_j\delta_{mn} - \frac{15}{32}\delta_{in}e_j e_m - \frac{15}{32}\delta_{im}e_j e_n\right. \\ &\left. + \frac{9}{32}\delta_{ij}e_n e_m - \frac{3}{32}\delta_{ij}\delta_{mn}\right) + \frac{1}{16}\varepsilon_2^3\left(\delta_{ij} - 3e_i e_j\right)\delta_{nm} + \frac{27}{256}\varepsilon_1\varepsilon_2^2\left(\delta_{ij} + 7e_i e_j\right)\delta_{nm} + O(5)\Bigg]; \end{aligned}$$

$$T^{11}_{ijmn} = \frac{3}{16}\pi r_1^4\left[\varepsilon_1^2\varepsilon_2\varepsilon_{ijl}e_l\delta_{mn} + O(5)\right],$$

$$T_{ijmn}^{22} = -\frac{3}{16}\pi r_2^4\left[\varepsilon_2^2\varepsilon_1\varepsilon_{ijl}e_l\delta_{mn} + O(5)\right],$$

$$T_{ijmn}^{12} = \frac{1}{2}\pi r_1^3 r_2\left[\varepsilon_2^2\varepsilon_{ijl}e_l\delta_{mn} + O(4)\right],$$

$$T_{ijmn}^{21} = \frac{1}{2}\pi r_2^3 r_1\left[\varepsilon_1^2\varepsilon_{ijl}e_l\delta_{mn} + O(4)\right].$$

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